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ON THE GROUPS $\mathrm{SL}_2(\mathbb{Z}[x])$ AND $\mathrm{SL}_2(k[x,y])$

BY

FRITZ GRUNEWALD

Fakultät für Mathematik, Universität Bonn
Beringstrasse 4, W-5300 Bonn 1, Germany

AND

JENS MENNICKE

Fakultät für Mathematik, Universität Bielefeld
W-4800 Bielefeld, Germany

AND

LEONID VASERSTEIN

Department of Mathematics, The Pennsylvania State University
University Park, PA 16802, USA

ABSTRACT

This paper studies free quotients of the groups $\mathrm{SL}_2(\mathbb{Z}[x])$ and $\mathrm{SL}_2(k[x,y])$, k a finite field. These quotients give information about the relation of the above groups to their subgroups generated by elementary or unipotent elements.

1. Introduction

The group theoretic structure of the special linear group $\mathrm{SL}_n(F)$ over a commutative field F is very well understood. The book of Dieudonné [D] for example contains a wealth of material on this subject. For a general associative and commutative ring R the groups $\mathrm{SL}_n(R)$ are much more difficult to study and there is no hope of obtaining results which are as complete as in the case of a field. Here we are concerned with rings as simple as the polynomial rings $R = \mathbb{Z}[x]$ or $k[x,y]$ where k is a finite field. For these rings the normal subgroups of $\mathrm{SL}_n(R)$ have

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been determined for $n \geq 3$ [V4]. As this paper will show the lattice of normal subgroups is vastly more rich for $n = 2$.

An approach often followed in the study of the groups $\mathrm{SL}_n(R)$ is to introduce the subgroups $E_n(R) \leq \mathrm{SL}_n(R)$ generated by the elementary matrices over R. The groups $E_n(R)$ and, in particular, $E_2(R)$ are easier to understand than the group $\mathrm{SL}_n(R)$. They can for example be studied by the methods of Cohn's paper [C]. In this approach it is important to control the difference between $E_n(R)$ and $\mathrm{SL}_n(R)$. Of course in case R is a field these two groups are equal. Let us describe what is known if R is a polynomial ring

$$R = A[x_1, \ldots, x_m]$$

over a ring A. Then

(1.1) for $n \geq 3$ the group $E_n(R)$ is normal in $\mathrm{SL}_n(R)$.

(1.2) for $n \geq 3$ and A euclidean we have $E_n(R) = \mathrm{SL}_n(R)$.

Actually, the result (1.1) is true even for any commutative ring R and is proved in [Su], see also [V4]; (1.2) was proved by Suslin [Su].

Not much seems to be known in the cases left out in (1.1) or (1.2). For $n = 2$ and $R = \mathbb{Z}[x]$ or $k[x, y]$ it is known that $E_2(R) \neq \mathrm{SL}_2(R)$, [C]. In fact Cohn [C] proves that

$$\begin{pmatrix} 1+2x & -x^2 \\ 4 & 1-2x \end{pmatrix} \notin E_2(\mathbb{Z}[x]), \tag{1.3}$$

$$\begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix} \notin E_2(k[x,y]). \tag{1.4}$$

In the present paper we study in more detail the deviation of $\mathrm{SL}_2(R)$ from $E_2(R)$ where $R = \mathbb{Z}[x]$ or $k[x, y]$, k a finite field.

In this connection we mention the conjecture of Suslin that $\mathrm{SL}_2(R)$ is generated by unipotent matrices in case $R = \mathbb{Z}[x]$ or $k[x, y]$, k a finite field. Note that the examples of Cohn (1.3), (1.4) are both unipotent.

To describe our results we introduce the following notation. First of all we assume for this paper that all rings are commutative and have an identity 1.

Definition 1.1: Let R be a ring and $n \geq 2$ an integer. We put

$$E_n(R) = \langle \{1_n + ae_{ij} \mid 1 \leq i \neq j \leq n,\ a \in R\} \rangle.$$

Here $\langle S \rangle$ stands for the subgroup of a group generated by the subset S, and 1_n is the n-dimensional unit matrix. The $n \times n$ matrix e_{ij} has zeros everywhere, except for the crossing of the i-th row and j-th column where there is a 1. We furthermore define:

$$U_n(R) = \langle \{g \in \mathrm{SL}_n(R) \mid g \text{ is unipotent}\} \rangle.$$

We write $\hat{E}_n(R)$ for the normal subgroup of $\mathrm{SL}_n(R)$ generated by $E_n(R)$.

We have $E_n(R) \leq \hat{E}_n(R) \leq U_n(R) \leq \mathrm{SL}_n(R)$. $U_n(R)$ is a normal subgroup of $\mathrm{SL}_n(R)$. Let

$$F_m$$

always denote the free group on m symbols. We prove:

THEOREM 1.2: *Let $m \geq 1$ be an integer. Then there is a surjective homomorphism*

$$\mathrm{SL}_2(\mathbb{Z}[x])/U_2(\mathbb{Z}[x]) \to F_m.$$

Theorem 1.2 shows that Suslin's conjecture is far from true for $R = \mathbb{Z}[x]$. In the following proposition we give an explicit example of an element which is not annihilated by a homomorphism such as in Theorem 1.2.

PROPOSITION 1.3: *Let*

$$g = \begin{pmatrix} 182x + 9 & -312x^2 + 5x + 1 \\ 98 & -168x + 11 \end{pmatrix}.$$

Then $g \in \mathrm{SL}_2(\mathbb{Z}[x])$ and there is a homomorphism

$$\mathrm{SL}_2(\mathbb{Z}[x])/U_2(\mathbb{Z}[x]) \to \mathbb{Z}$$

which maps g to $1 \in \mathbb{Z}$.

More examples of this type are constructed in section 3 of this paper. We also prove:

THEOREM 1.4: *Let k be a finite field and $m \geq 1$ an integer. Then there is a surjective group homomorphism*

$$\mathrm{SL}_2(k[x,y])/U_2(k[x,y]) \to F_m.$$

Note that every homomorphism from $\mathrm{SL}_2(k[x,y])$ to F_m factors through $\mathrm{SL}_2(k[x,y])/U_2(k[x,y])$ since $U_2(k[x,y])$ is generated by p-torsion elements.

We shall complete the story by giving certain matrices in $\mathrm{SL}_2(k[x,y])$ which have nontrival images in free quotients. To do this we make the following definition.

Definition 1.5: Let k be a finite field and r, $s \in k$. Let

$$M(r,s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with:

$$\begin{aligned} a &= 1 - rx + r^2x^2 + (-sx + 2rsx^2)y + s^2x^2y^2, \\ b &= r^3 + 3r^2sy + 3rs^2y^2 + s^3y^3, \\ c &= x^3, \\ d &= 1 + rx + sxy. \end{aligned}$$

Note that $M(r,s) \in \mathrm{SL}_2(k[x,y])$ for every r, $s \in k$. We prove:

PROPOSITION 1.6: *Let k be a finite field of characteristic not 2 or 3 and r, $s \in k$. Assume that:*

(i) *$r \neq 0, 1, -1$ and $s \neq 0$,*
(ii) *$2rs$ is not a square in k,*
(iii) *$-s$ is a square in k.*

Then there is a homomorphism

$$\mathrm{SL}_2(k[x,y])/U_2(k[x,y]) \to \mathbb{Z}$$

which maps $M(r,s)$ to 1.

The free quotients of $\mathrm{SL}_2(R)$, $R = \mathbb{Z}[x]$ or $k[x,y]$ are obtained by studying certain quotient rings $\mathcal{O}$ of R and the induced homomorphisms $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathcal{O})$. The situation will be so arranged that $\mathrm{SL}_2(\mathcal{O})$ has free quotients. For $R = \mathbb{Z}[x]$ we use certain rings of imaginary quadratic integers as $\mathcal{O}$. The group $\mathrm{SL}_2(\mathcal{O})$ then acts on the 3-dimensional hyperbolic space. In section 3 we give certain details on this action. For $R = k[x,y]$ we use the rings $\mathcal{O}$ of integers in hyperelliptic function fields over k. Then $SL_2(\mathcal{O})$ acts on a Tits building. We give a general method which allows us to construct examples also in the case the characteristic of k is 2 or 3. Section 4 even contains explicit (finite) sets of matrices in $SL_2(k[x,y])$ which can be mapped onto the generators of free quotients of rank bigger than 1. We shall now describe our results on the map $SL_2(R) \to SL_2(S)$ induced by a ring homomorphism $R \to S$.

Definition 1.6: Let R, S be two rings and $\varphi\colon R \to S$ a ring homomorphism. We write

$$\varphi_n\colon SL_n(R) \to SL_n(S)$$

for the homomorphism which applies φ to the entries of a matrix.

If $\varphi\colon R \to S$ is a surjective ring homomorphism, it is in general very difficult to understand the images of the φ_n. Only if $SL_n(S)$ is generated by elementary matrices (i.e. $SL_n(S) = E_n(S)$) it is clear that φ_n is surjective. If $R = \mathbb{Z}[x]$ or $k[x,y]$ (k a field) and $\varphi\colon R \to S$ is a surjective ring homomorphism, (1.2) implies that φ_n is surjective for $n \geq 3$. For $n = 2$ we prove:

THEOREM 1.7:

(1) *Let $\varphi\colon \mathbb{Z}[x] \to S$ be a surjective ring homomorphism. Then* $\mathrm{Im}(\varphi_2)$ *is a normal subgroup of finite index in* $SL_2(S)$ *with abelian quotient.*
(2) *Let k be a field and $\varphi\colon k[x,y] \to S$ be a surjective ring homomorphism. Then φ_2 is surjective.*

In general it was quite difficult to find preimages under φ_2 for specific elements. Consider the following special case. Let $\mathcal{O}$ be the full ring of integers in the field $\mathbb{Q}(\sqrt{-5})$, that is $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$. The map

$$\varphi\colon \begin{cases} \mathbb{Z}[x] \to \mathcal{O} \\ p(x) \mapsto p(\sqrt{-5}) \end{cases}$$

is a surjective ring homomorphism. It will be clear from the following that φ_2 is surjective. Despite this it is very difficult to find (even with a computer) a preimage of

$$g = \begin{pmatrix} -\sqrt{-5} & 2 \\ 2 & \sqrt{-5} \end{pmatrix} \in SL_2(\mathcal{O}).$$

Putting indeterminate entries for a preimage of g so that all entries have degree $\leq m$ leads to a system of 4 diophantine equations in $4m$ unknowns. In connection with these equations one often wonders whether g might have a preimage all of whose entries have degree $\leq 2, 3, \ldots$. We prove:

THEOREM 1.8:

(1) *Let $\mathcal{O}$ be a full ring of S-arithmetic integers in an algebraic number field. Let $\varphi\colon \mathbb{Z}[x] \to \mathcal{O}$ be a surjective ring homomorphism. There is an integer d so that every $h \in \mathrm{Im}(\varphi_2)$ has a preimage h so that all entries of g have degree $\leq d$.*

(2) *Let F be a function field of one variable over a finite field k. Let S be a finite set of prime divisors of F containing at least two elements of coprime degrees. Let*

$$\mathcal{O}_S = \bigcap_{P \notin S} \mathcal{O}_P$$

be the full ring of S-arithmetic integers. Let furthermore

$$\varphi\colon k[x,y] \to \mathcal{O}_S$$

be a surjective ring homomorphism. Assume that $\mathcal{O}_S$ is of rank n over the subring $\varphi(k[x])$. Then every element of $\mathrm{SL}_2(\mathcal{O}_S)$ has a priemage in $\mathrm{SL}_2(k[x,y])$ under φ_2 with all entries of y-degree $\leq 8 \cdot n$.

If the ring $\mathcal{O}$ in Theorem 1.8.1 has a unit of infinite order and if the generalized Riemann hypothesis is valid, then the result of Cooke and Weinberger, [C,W], implies that every element in $\mathrm{SL}_2(\mathcal{O})$ is the product of at most n elementary matrices, where n is some universal constant. From this our result follows; moreover the d may (under the above hypothesis) be chosen only depending on the dimension over $\mathbb{Q}$ of the field of fractions of $\mathcal{O}$.

In section 5 we also give certain information on the size of the groups

$$U_2(R)/\hat{E}_2(R)$$

when $R = \mathbb{Z}[x]$ or $k[x,y]$.

Finally in section 6 we study the Bass stable range of polynomial rings. Following [V,S] we use $\mathrm{sr}(R)$ for the stable range of a ring R. A consequence of our considerations is that

$$\mathrm{sr}(\mathbb{Z}[x]) = 3.$$

This is well known by [V,S]. An advantage of our approach is that we are able to construct explicit unimodular vectors which are not stable (see section 6 for the definition). A particular example is contained in the following proposition.

PROPOSITION 1.9: *The vector*

$$(21+4x,\ 12,\ x^2+20)$$

is unimodular but is not stable in $\mathbb{Z}[x]^3$. *That is, there are no* $a, b \in \mathbb{Z}[x]$ *so that* $(21+4x+a(x^2+20),\ 12+b(x^2+20))$ *is unimodular.*

Note in this connection that $\mathrm{sr}(k[x,y]) = 2$ whenever k is a finite field, [V,S].

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2. Symplectic matrices and the image of φ_2

Here we give the method by which we shall describe the image of

$$\varphi_2\colon \mathrm{SL}_2(R) \to \mathrm{SL}_2(S)$$

for certain ring homomorphisms $\varphi\colon R \to S$.

First of all, we introduce some notation concerning the symplectic groups. For a natural number n we define

$$J_n = \sum_{i=1}^{n} e_{2i-1,2i} - \sum_{i=1}^{n} e_{2i,2i-1}$$

to be the standard $2n \times 2n$ alternating matrix

$$J_n = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

The symplectic group over the ring R is defined as

$$\mathrm{Sp}_{2n}(R) = \{g \in M_{2n}(R) \mid g^t J_n g = J_n\}.$$

Here g^t stands for the transpose of the matrix g. To define the elementary subgroup we introduce the following permutation of $\mathbb{Z}$:

$$(2i)' = 2i-1, \quad (2i-1)' = 2i.$$

For an element $a \in R$ and natural numbers $1 \leq i \neq j \leq 2n$ we define the following (elementary symplectic) $2n \times 2n$ matrix:

$$SE_{ij}(a) = \begin{cases} 1_{2n} + ae_{ij} & \text{if } i = j' \\ 1_{2n} + ae_{ij} - (-1)^{i+j} ae_{j'i'} & \text{if } i \neq j' \end{cases}.$$

If $\mathcal{A} \subseteq R$ is an ideal we define

$$\mathrm{Ep}_{2n}(\mathcal{A}) = \langle \{ SE_{ij}(a) | \, a \in \mathcal{A} \} \rangle.$$

Let furthermore

$$\mathrm{Ep}_{2n}(R, \mathcal{A})$$

be the normal subgroup of $\mathrm{Ep}_{2n}(R)$ generated by $\mathrm{Ep}_{2n}(\mathcal{A})$. Note that $\mathrm{Ep}_{2n}(R)$ is always normal in $\mathrm{Sp}_{2n}(R)$ for $n \geq 2$.

We also define for a natural number n

$$Um_n(R, \mathcal{A}) = \left\{ (r_1, \dots, r_n) \in R^n | \begin{array}{c} R = r_1 R + \cdots + r_n R \text{ and} \\ (r_1, \dots, r_n) \equiv (0, \dots, 0, 1) \bmod \mathcal{A} \end{array} \right\}.$$

We also put $Um_n(R) = Um_n(R, R)$.

The group $\mathrm{Ep}_{2n}(R, \mathcal{A})$ acts by right multiplication on $Um_{2n}(R, \mathcal{A})$. We shall need some results on this action in the sequel. For our purposes polynomial rings $R = A[x_1, \dots, x_m]$ are of interest where A is **locally principal**, that is the localization A_μ is a principal ideal ring for every maximal ideal μ of A.

LEMMA 2.1: *Let A be a locally principal ring, and let $m \geq 0$ and $n \geq 2$ be integers. Let furthermore $\mathcal{A} \subseteq A[x_1, \dots, x_m]$ be an ideal. Assume that $m \leq 2n - 3$. Then the group $\mathrm{Ep}_{2n}(A[x_1, \dots, x_m], \mathcal{A})$ acts transitively on $Um_{2n}(A[x_1, \dots, x_m], \mathcal{A})$.*

Proof: Since A is locally principal the Krull dimension of $R = A[x_1, \dots, x_m]$ is $\leq m+1$. So by Bass' result the stable range of R is $\leq m+2$. We infer from [B1] chapter 5.3 that the stable range of R relative to the ideal $\mathcal{A}$ is $\leq m+2$. From [V,S] Theorem 7.3 we obtain the result. ∎

Note that in case $\mathcal{A} = A[x_1, \dots, x_m]$ we have a much sharper result [G,M,V].

For natural numbers $n \leq m$ we consider the following imbeddings:

$$\psi: \begin{cases} \mathrm{Sp}_{2n}(R) \to \mathrm{Sp}_{2m}(R) \\ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1_{2m-2n} \end{pmatrix} \end{cases}.$$

Using these maps the symplectic K_1-group is defined as

$$K_1\mathrm{Sp}(R) = \lim_{n\to\infty} \mathrm{Sp}_{2n}(R)/\lim_{n\to\infty} \mathrm{Ep}_{2n}(R).$$

We write

$$\mathrm{WSp}_{2n}\colon \mathrm{Sp}_{2n}(R) \to K_1\mathrm{Sp}(R)$$

for the resulting Whitehead homomorphism. For $n \geq 2$ this map induces homomorphisms:

$$\tilde{\mathrm{W}}\mathrm{Sp}_{2n}\colon \mathrm{Sp}_{2n}(R)/\mathrm{Ep}_{2n}(R) \to K_1\mathrm{Sp}(R).$$

Our general result is:

PROPOSITION 2.2: *Let A be a locally principal ring, and let $\varphi\colon A[x] \to B$ be a surjective ring homomorphism. Assume that*

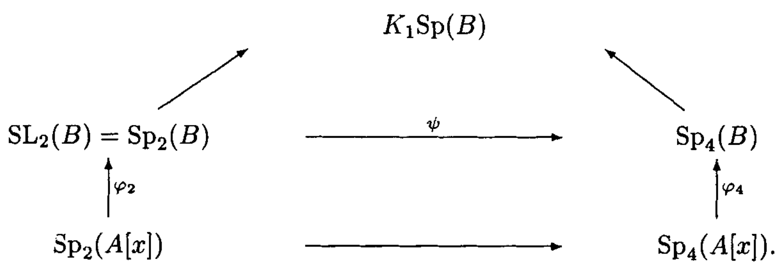
(1) $K_1\mathrm{Sp}(A) = 1$,

(2) *the map $\tilde{W}\mathrm{Sp}_4\colon \mathrm{Sp}_4(B)/\mathrm{Ep}_4(B) \to K_1\mathrm{Sp}(B)$ is injective.*

Then:

$$\varphi_2(\mathrm{SL}_2(A[x])) = \ker(\mathrm{WSp}_2).$$

Proof: Consider the diagram



If $g \in \mathrm{Sp}_2(B)$ is an element with $\mathrm{WSp}_2(g) = 1$, then by our assumptions

$$\psi(g) \in \mathrm{Ep}_4(R),$$

and

$$\psi(g) \in \mathrm{Im}(\varphi_4).$$

Let $h \in \mathrm{Sp}_4(A[x])$ be a matrix with $\varphi_4(h) = \varphi(g)$. The last row of h lies in $Um_4(A[x]), \mathcal{A})$ where $\mathcal{A}$ is defined to be the kernel of φ. From Lemma 2.1 and an elementary computation we infer that we can choose an $h_1 \in \mathrm{Ep}_4(A[x], \mathcal{A})$ so that

$$h_1 \cdot h = \begin{pmatrix} a_1 & a_2 & b_1 & 0 \\ a_3 & a_4 & b_2 & 0 \\ 0 & 0 & 1 & 0 \\ c_3 & c_4 & d_2 & 1 \end{pmatrix}$$

with appropriate choices of entries. From the definition of elementary symplectic matrices it is then clear that there is an $h_2 \in \mathrm{Ep}_4(A[x], \mathcal{A})$ with

$$h_2 \cdot h_1 \cdot h \in \varphi(\mathrm{Sp}_2(A[x])).$$

Choose $g_1 \in \mathrm{Sp}_2(A[x])$ with $h_2 h_1 h = \psi(g_1)$. It then follows that $\varphi_2(g_1) = g$. If conversely for $g \in \mathrm{Sp}_2(B)$ there is an element $g_1 \in \mathrm{Sp}_2(A[x])$ with $\varphi_2(g_1) = g$, then by [G,M,V] Corollary 1.4 we have

$$\psi(g_1) \in \mathrm{Ep}_4(A[x]).$$

This implies $\mathrm{WSp}_2(g) = 1$. ∎

Proposition 2.2 is similar to Lemma 17.1 of [V,S]. We have included a proof since we shall need it in the sequel.

Proof of Theorem 1.7: (1) Let $\varphi\colon \mathbb{Z}[x] \to S$ be a surjective ring homomorphism with kernel $\mathcal{A}$. By Theorem 2.2 we have to prove that $K_1\mathrm{Sp}(S)$ is finite. If $\mathcal{A}$ is not principal then S is finite and the result follows. Let now $\mathcal{A} = a \cdot \mathbb{Z}[x]$ with a nonzero polynomial a. We infer from [V2] that

$$K_1\mathrm{Sp}(S) = \mathrm{SK}_1(S).$$

The polynomial a can be factorized as

$$a = l \cdot P_1^{e_1} \cdots P_r^{e_r} = l \cdot a'$$

where $l, e_1, \ldots, e_r \in \mathbb{N}$ and the $P_i \in \mathbb{Z}[x]$ being irreducible in $\mathbb{Z}[x]$. Put

$$S' = \mathbb{Z}[x]/a' \cdot \mathbb{Z}[x], \quad S'' = \mathbb{Z}[x]/l \cdot \mathbb{Z}[x].$$

We identify S in the obvious way with a subring of $R = S' \times S''$. The ideal

$$\mathcal{B} = S' \cdot l + S'' \cdot a' \subseteq R$$

is common to R and S and has finite index in S. Hence ([V,S]) $SK_1(S)$ is a quotient of $SK_1(S, \mathcal{B}^2)$. On the other hand $SK_1(S, \mathcal{B}^2)$ is a quotient of $SK_1(R, \mathcal{B})$ by [V,S], §16. We furthermore have

$$SK_1(R, \mathcal{B}) = SK_1(S', \mathcal{B}') \times SK_1(S'', \mathcal{B}'')$$

where $\mathcal{B}' = S' \cap \mathcal{B}'$ and $\mathcal{B}'' = S'' \cap \mathcal{B}'$.

This leaves us with proving the finiteness of $SK_1(S', \mathcal{B}')$ and $SK_1(S'', \mathcal{B}'')$. In fact we have $SK_1(S'', \mathcal{B}'') = 1$ by [V,S], §16.

In the remaining case we have

$$SK_1(S', \mathcal{B}') \cong SK_1(\mathcal{O}_1, \mathcal{B}_1) \times \cdots \times SK_1(\mathcal{O}_r, \mathcal{B}_r)$$

where $\mathcal{O}_i = \mathbb{Z}[x]/P_i \cdot \mathbb{Z}[x]$ and the $\mathcal{B}_i$ are the projections of the ideal $\mathcal{B}'$ in $\mathcal{O}_i$. Here we have used the fact that $SK_1(R, \mathcal{B}) = SK_1(R/\operatorname{rad}(R), \bar{\mathcal{B}})$ where $\operatorname{rad}(R)$ is the Jacobson radical of R and $\bar{\mathcal{B}}$ is the image of the ideal $\mathcal{B}$ in $R/\operatorname{rad}(R)$, (see [V,S]). The rings $\mathcal{O}_i$ are orders in Dedekind rings of arithmetic types and the finiteness of $SK_1(\mathcal{O}_i, \mathcal{B}_i)$ follows from [B,M,S].

(2) We infer from [V,S], §16 that $K_1\mathrm{Sp}(S) = 1$. Using Theorem 2.2 the result is proved. ∎

The rest of this section will be devoted to the proof of Theorem 1.8. We have to start with general remarks on the arithmetic of polynomial rings.

LEMMA 2.3: *Let A be a principal ideal domain. There is a function*

$$f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$$

so that given $u = (a_1, \ldots, a_n) \in Um_n(A[x])$ with $\deg(a_i) \le k \in \mathbb{N} \cup \{0\}$ for $i = 1, \ldots, n$, there are $\lambda_1, \ldots, \lambda_n \in A[x]$ with $\deg(\lambda_i) \le f(k)$ for $i = 1, \ldots, n$ satisfying

$$\lambda_1 a_1 + \cdots + \lambda_n a_n = 1.$$

Proof: Let K be the field of fractions of A. Using the euclidean algorithm in $K[x]$, we find $\mu_1, \ldots, \mu_n \in A[x]$ of bounded degree and $\alpha \in A$ with $\alpha \neq 0$ so that

$$\mu_1 a_1 + \cdots + \mu_n a_n = \alpha.$$

If α is a unit in A we are done. If not write

$$\alpha = \prod_{i=1}^{l} \pi_i^{e_i}$$

for the factorization of α into distinct prime elements π_i. Using the euclidean algorithm in $A/\pi_i A$ we find polynomials $\nu_{ij} \in A[x]$ of bounded degree so that

$$\sum_{j=1}^{n} \nu_{ij} a_j = 1 + \pi_i Q_i \tag{2.1}$$

for some $Q_i \in A[x]$. Note that $1 + \pi_i Q_i$ is a unit in $A/\pi_i^{e_i} A$ with an inverse of bounded degree. We multiply the equations (2.1) by these inverses. Using the chinese remainder theorem we find $\nu_1, \ldots, \nu_n \in A[x]$ of bounded degree with

$$\nu_1 a_1 + \cdots + \nu_n a_n = 1 + \alpha Q$$

for some $Q \in A[x]$. ∎

PROPOSITION 2.3: *Let A be a principal ideal domain. Then there is a function*

$$g \colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$$

so that given $(a_1, a_2, a_3, a_4) \in Um_4(A[x])$ with $\deg(a_i) \leq k \in \mathbb{N} \cup \{0\}$ for $i = 1, \ldots, 4$ there are $\lambda_2, \lambda_3, \lambda_4 \in A[x]$ with $\deg(\lambda_2), \deg(\lambda_3), \deg(\lambda_4) \leq g(n)$ so that

$$(a_2 + \lambda_2 a_1,\ a_3 + \lambda_3 a_1,\ a_4 + \lambda_4 a_1) \in Um_3(A[x]).$$

Proof: If $a_3 = 0$ then we add a_1 to the third component and the result follows. Assume now $a_3 \neq 0$.

Let K be the field of fractions of A. We shall first prove the result for the ring $K[x]$. We write

$$a_3 = \prod_{i=1}^{k} P_i^{e_i}$$

for the factorization of a_3 into distinct irreducible polynomials P_i. We order the P_i so that exactly the $P_1, \ldots, P_r$ divide a_4. By solving suitable congruences modulo $P_1, \ldots, P_r, P_{r+1}, \ldots, P_k$ we find polynomials $\lambda, \mu \in A[x]$ of bounded degree so that a_3 and $a_4 + \lambda a_2 + \mu a_1$ are coprime in $K[x]$. Since the operation

$$(a_2, a_3, a_4) \to (a_2, a_3, a_4 + \lambda a_2)$$

is a linear automorphism of $A[x]^3$ we may furthermore assume that there are $\mu_3, \mu_4 \in A[x]$ and $\alpha \in A$ with $\alpha \neq 0$ so that

$$\mu_3 a_3 + \mu_4 a_4 = \alpha. \tag{2.2}$$

If α is a unit in A we are finished, if not we write

$$\alpha = \pi_1^{f_1} \cdots \pi_s^{f_s}$$

for the factorization of α into distinct prime elements of A. Consider now the problem over the fields $A/\pi_i A$. By solving suitable congruences in $A/\pi A[x]$ we find an $\nu \in A[x]$ of bounded degree so that the ideal

$$\mathcal{A} = (a_2 + \nu a_1)A[x] + a_3 A[x] + a_4 A[x]$$

contains a polynomial

$$P + 1 + b_1 x + \cdots + b_l x^l$$

with $\pi_i | \ b_j$ for $j = 1, \ldots, l$. Notice that the image of P is a unit in the ring $A/\pi_i^{f_i}[x]$. By the chinese remainder theorem the ideal $\mathcal{A}$ contains a polynomial which is a unit in $A/\alpha A[x]$. Since $\mathcal{A}$ contains a_3, a_4 and hence by (2.2) also α it contains 1. ∎

We shall recall now the stable range defined by Bass. Let R be a commutative ring and $r \geq 2$ an integer. A unimodular vector

$$(a_1, \ldots, a_r) \in Um_r(R)$$

is called **stable** if there are $\lambda_2, \ldots, \lambda_r \in R$ so that

$$(a_2 + \lambda_2 a_1, \ldots, a_r + \lambda_r a_1) \in Um_{r-1}(R).$$

The **stable range** $\mathrm{sr}(R)$ of R is the infimum over all natural numbers $r \geq 1$ which have the property that every vector in $Um_{r+1}(R)$ is stable. Note that this definition differs by 1 from the definition of Bass [B1]. It coincides with the definition of [V,S].

An important result of Bass [B1] is that the stable range of R is less than or equal to $d + 1$ if the maximal spectrum of R is a noetherian space of dimension d. Proposition 2.4 is a version of this result for principal ideal domains with a bound on degrees. An obvious consequence is:

PROPOSITION 2.4: *Let A be a principal ideal domain. Put $A' = A^{\mathbb{N}}$. Then*

$$\mathrm{sr}(A'[x]) \leq 3.$$

Notice that Bass' result cannot be used here since A' is not noetherian.

We shall generalize the transitivity results in [B2] section 4 for the action of the elementary group in SL_n to the symplectic case.

PROPOSITION 2.5: *Let R be a commutative ring with $\mathrm{sr}(R) \le m$ and $\mathcal{A} \subseteq R$ an ideal. Let $u \in Um_{2n}(R, \mathcal{A})$ for some $n \in \mathbb{N}$ with $2n \ge m$. Then there is a $\gamma \in \mathrm{Ep}_{2n}(R, \mathcal{A})$ with*

$$\gamma u^t = (0, \ldots, 0, 1)^t.$$

Proof: For notational reason we shall give the proof for $n = 2$ and $m = 3$. It will be obvious how to do the general case. Let

$$u = (a_1, a_2, a_3, a_4) \in Um_4(R).$$

Choose $\lambda_2, \lambda_3, \lambda_4 \in R$ so that

$$(a_2 + \lambda_2 a_1,\ a_3 + \lambda_3 a_1,\ a_4 + \lambda_4 a_1) \in Um_3(R).$$

By the result 4, Proposition 3.2 of Bass [B2], the $\lambda_2, \lambda_3, \lambda_4$ can be chosen in $\mathcal{A}$. We multiply u^t by $g_1 = SE_{21}(-\lambda_3\lambda_4 a_1 + \lambda_2 a_1)SE_{31}(\lambda_3)SE_{41}(\lambda_4)$ to obtain

$$\begin{aligned} g_1 u^t =& (a_1,\ a_2 - \lambda_4(a_3 + \lambda_3 a_1) + \lambda_3(a_4 + \lambda_4 a_1) + \lambda_2 a_1,\ a_3 + \lambda_3 a_1,\ a_4 + \lambda_4 a_1)^t \\ =& (a_1,\ a_2',\ a_3',\ a_4')^t = u_1^t. \end{aligned}$$

So we may further assume that $(a_2',\ a_3',\ a_4') \in Um_3(R)$. We choose $\mu_2, \mu_3, \mu_4 \in R$ with

$$\mu_2 a_2' + \mu_3 a_3' + \mu_4 a_4' = 1.$$

We put

$$t = 1 - a_1 - a_4'.$$

Note that $t \in \mathcal{A}$. We multiply u_1^t now by

$$g_2 = SE_{12}(-t\mu_3)SE_{14}(1)SE_{12}(-t^2\mu_3\mu_4)SE_{14}(t\mu_4)SE_{13}(t\mu_3)SE_{12}(t\mu_2)$$

to obtain

$$\begin{aligned} g_2 u_1^t =& (1,\ a_2',\ a_3' - a_2' - t\mu_4 a_2',\ a_4 + t\mu_3 a_2')^t \\ =& (1,\ a_2',\ a_3'',\ a_4'')^t = u_2^t. \end{aligned}$$

Note that $a_3'' \in \mathcal{A}$ and $a_4'' = 1 + b$ for some $b \in \mathcal{A}$. We put

$$g_3 = SE_{12}(-a_2'' - b a_3'')SE_{34}(-a_3'')SE_{41}(-b)$$

and obtain

$$g_3 u_2^t = (1,\ 0,\ 0,\ 1)^t.$$

We conclude by noting that

$$SE_{14}(-1)g_3g_2g_1 \in \mathrm{Ep}_4(R,\mathcal{A}). \quad \blacksquare$$

The preceding Proposition is proved in [V,S] for $\mathcal{A} = R$ by using the analogous result of Bass [B2] for SL_n. We shall proceed by stating a technical result which will imply Theorem 1.8.

PROPOSITION 2.6: *Let A be a principal ideal domain and $\mathcal{A} \subseteq A[x]$ an ideal. Then there is a function*

$$h\colon \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$$

so that given $u \in Um_4(A[x],\mathcal{A})$ with all entries of degree $\leq k$, there is a $g \in \mathrm{Ep}_4(A[x],\mathcal{A})$ with all entries bounded by $h(k)$ so that

$$gu^t = (0,\ 0,\ 0;\ 1)^t.$$

Proof: Put $A' = A^{\mathbb{N}}$ and define

$$\varphi\colon A'[x] \to (A[x])^{\mathbb{N}}$$

to be the obvious ring homomorphism. Write $\bar{\mathcal{A}} = \mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}' = \varphi^{-1}(\bar{\mathcal{A}})$. $\mathcal{A}'$ is an ideal in $\mathcal{A}'[x]$. Order the elements in $Um_4(A[x])$ which are of degree $\leq k$ to obtain a vector $u \in Um_4((A[x])^{\mathbb{N}}, \bar{\mathcal{A}})$. u being componentwise in the image of φ, put $u_1 = \varphi^{-1}(u)$. By Lemma 2.3 u_1 is in $Um_4(A'[x],\mathcal{A}')$. Applying Proposition 2.6 for $R = A'[x]$ and the ideal $\mathcal{A}'$ we conclude the result. $\blacksquare$

PROPOSITION 2.7: *Let A be a principal ideal domain with $K_1\mathrm{Sp}(A) = 1$. Let*

$$\varphi\colon A[x] \to B$$

be a surjective ringhomomorphism. Assume that there is a $n \in \mathbb{N}$ so that every $g \in \ker(\mathrm{WSp}_4) \leq \mathrm{Sp}_4(B)$ is a product of at most n elementary matrices. Then there is a $d \in \mathbb{N}$ so that every $h \in \operatorname{im}\varphi_2$ has a preimage in $\mathrm{SL}_2(A[x])$ with all entries of degree $\leq d$.

Proof: By our assumption we see that the Krull dimension of B is ≤ 2 so by [V5] we have $\mathrm{Ep}_4(B) = \ker(\mathrm{WSp}_4)$ and the assumptions of Proposition 2.2 are satisfied. By going through the proof and using Proposition 2.5 instead of Lemma 2.1 we get the result. $\blacksquare$

Proof of Theorem 1.8: (1) By the result of Tavgen [Ta] the hypothesis of Proposition 2.6 is valid for the arithmetic rings mentioned in the statement of the theorem.

(2) By the result of Queen [Q] every element in $\mathrm{SL}_2(\mathcal{O}_S)$ can be written as a product of 4 elementary matrices and a matrix

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

with $\varepsilon \in \mathcal{O}_S^*$, which then also can be written as a product of 4 elementary matrices. ∎

3. Free quotients of $\mathrm{SL}_2(\mathbb{Z}[x])$

To prove Theorem 1.2 and to construct several examples let $d > 0$ be an integer and put

$$\mathcal{O}_d = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

$\mathcal{O}_d$ is an order in the number field $\mathbb{Q}(\sqrt{-d})$.

Proof of Theorem 1.2: Let the integer $k \leq m + 1$ be given. It is proved in [G,S] that there is a $d \in \mathbb{N}$ and a surjective group homomorphism

$$\Delta \colon \mathrm{SL}_2(\mathcal{O}_d) \to F_k.$$

By the following it can be shown that there is a surjective homomorphism

$$\alpha \colon F_k \to F_{k-1}$$

so that

$$\Theta = \alpha \circ \Delta(U_2(\mathcal{O}_d)) = 1.$$

We only sketch here the topological argument which uses the construction of Δ in [G,S]. Every unipotent element g of $\mathrm{SL}_2(\mathcal{O}_d)$ stabilizes a cusp P in the boundary of hyperbolic 3-space. If this cusp P is not ∞, we may choose the base point Q for the construction of Θ near P. By continuity it follows that gQ is near P and we may infer $\Delta(g) = 1$. It remains to describe the image of the stabiliser of ∞ in F_k. The stabiliser of ∞ in $\mathrm{SL}_2(\mathcal{O}_d)$ is a finite extension of the elementary group

$$\left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathcal{O}_d\right\}.$$

By the construction of $\triangle$ its image is cyclic generated by

$$\Theta\left(\begin{pmatrix} 1 & \sqrt{-d} \\ 0 & 1 \end{pmatrix}\right).$$

If $l_1, \dots, l_k$ are the standard free generators of the fundamental group

$$\pi_1(B) = F_k$$

of the bouquet B of k circles, we find that

$$\triangle\left(\begin{pmatrix} 1 & \sqrt{-d} \\ 0 & 1 \end{pmatrix}\right) = l_1 \cdots l_k.$$

From this information we easily find the homomorphism α. An argument similar to the above is used in [G,S,1].

Let $\varphi\colon \mathbb{Z}[x] \to \mathcal{O}_d$ be the ring homomorphism which maps x to $\sqrt{-d}$.

The symplectic Whitehead group $K_1\mathrm{Sp}(\mathcal{O})$ is finite by [V,S], hence by Proposition 2.2 the image of

$$\Theta \circ \varphi_2\colon \mathrm{SL}_2(\mathbb{Z}[x]) \to F_m$$

has finite index and is therefore free of a rank which is bigger than or equal to m. ∎

Next we study the ring homomorphisms φ_2 for certain explicit ring homomorphisms $\varphi\colon \mathbb{Z}[x] \to \mathcal{O}_d$. This leads to certain explicit homomorphisms from $\mathrm{SL}_2(\mathbb{Z}[x])$ to $\mathbb{Z}$ and to a proof of Proposition 1.3.

The group $\mathrm{SL}_2(\mathcal{O}_d)$ acts via the embedding $\mathrm{SL}_2(\mathcal{O}_d) \subseteq \mathrm{SL}_2(\mathbb{C})$ discontinuously on three-dimensional hyperbolic space. In some cases fundamental domains are known [Sw], [G,H,M]. This leads to a presentation of $\mathrm{SL}_2(\mathcal{O}_d)$ and also to an understanding of the factor-group $\mathrm{SL}_2(\mathcal{O}_d)/U_2(\mathcal{O}_d)$. We demonstrate this in several cases.

To prove, for example, Proposition 1.3 it is then necessary to find preimages of certain elements under the homomorphism

$$\mathrm{SL}_2(\mathbb{Z}[x]) \to \mathrm{SL}_2(\mathcal{O}_d)/U_2(\mathcal{O}_d).$$

This is in general very difficult and we really do not know how to do this in a practical way. We then proceeded the other way around. We took certain elements in $\mathrm{SL}_2(\mathbb{Z}[x])$ known to us and studied their images in $\mathrm{SL}_2(\mathcal{O}_d)/U_2(\mathcal{O}_d)$. The elements in $\mathrm{SL}_2(\mathbb{Z}[x])$ were constructed by the following simple observation.

Observation 3.1: Let R be a ring and $\alpha_0, \alpha_1, \beta_0, \beta_1, \delta_0, \delta_1, \gamma_0, r \in R$. Then

$$\begin{pmatrix} \alpha_0 + r\alpha_1\gamma_0 x & \beta_0 + \beta_1 x + r\alpha_1\delta_1 x^2 \\ r\gamma_0^2 & \delta_0 + r\gamma_0\delta_1 x \end{pmatrix} \tag{3.1}$$

is in $\mathrm{SL}_2(\mathbb{Z}[x])$ if the equations

$$\alpha_0\delta_1 + \alpha_1\delta_0 - \beta_1\gamma_0 = 0 \quad \text{and} \quad \alpha_0\delta_0 - r\beta_0\gamma_0^2 = 1 \tag{3.2}$$

are satisfied. In fact if R is a principal ideal domain then every

$$\begin{pmatrix} a_0 + a_1 x & b_0 + b_1 x + b_2 x^2 \\ c_0 & d_0 + d_1 x \end{pmatrix} \in \mathrm{SL}_2(R[x])$$

where the $a_0, a_1, b_0, b_1, c_0, d_0, d_1$ are in R can be written in the parametrized form (3.1). The equations (3.2) are simple enough to find many solutions directly.

Example 1: We put $\omega = \sqrt{-5}$ and

$$\mathcal{O}_5 = \mathbb{Z} + \mathbb{Z}\omega \subseteq \mathbb{Q}(\omega).$$

The ring $\mathcal{O}_5$ is the ring of integers in $\mathbb{Q}(\omega)$. We consider the ring homomorphism

$$\varphi: \begin{cases} \mathbb{Z}[x] \to \mathcal{O}_5 \\ x \mapsto \omega \end{cases}.$$

PROPOSITION 3.2: *Let φ be defined as above. Then the images under φ_2 of the matrices*

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ h_2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ h_3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},\ h_4 = \begin{pmatrix} 1+2x & x^2 \\ -4 & 1-2x \end{pmatrix},$$
$$h_5 = \begin{pmatrix} 6045x + 218 & 3224x^2 + 144x + 1 \\ 6975 & 3720x + 32 \end{pmatrix},$$
$$h_6 = \begin{pmatrix} 16211x + 591 & 1677x^2 + 116x + 2 \\ 10933 & 1131x + 37 \end{pmatrix}$$

generate $\mathrm{SL}_2(\mathcal{O}_5)$. There is a homomorphism

$$\eta: \mathrm{SL}_2(\mathbb{Z}[x])/\hat{E}_2(\mathbb{Z}[x]) \to \mathbb{Z}$$

with $\eta(h_i) = 0$ for all $i = 1, \dots, 5$ and $\eta(h_6) = 1$.

Proof: By Swan [Sw], the group $\mathrm{SL}_2(\mathcal{O}_5)$ is generated by the matrices:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix},\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} -\omega & 2 \\ 2 & \omega \end{pmatrix},$$
$$C = \begin{pmatrix} -\omega - 4 & -2\omega \\ 2\omega & \omega - 4 \end{pmatrix}.$$

By an analysis of the fundamental domain given by Swan we find:

$$\varphi_2(h_4) = BTB^{-1},$$
$$\varphi_2(h_5) = UAU^{-1}A^2BTB^{-1}A^2TA^{-1}BTB^{-1}AU^{-1}T^{-1}A^{-1}TU^{-1}AT^5A^{-1}U^2B,$$
$$\varphi_2(h_6) = U^2BU^{-1}T^{-3}B^{-1}T^2UBU^{-1}CB^{-1}AT^{-1}BU^{-1}T^{-1}A^{-1}UB^{-1}U^{-1} AT^{-1}A^{-1}UB^{-1}U^{-1}A^{-1}T^2U^{-2}AU^2A^{-1}.$$

From these formulas and the relations given in [Sw] the result follows. ∎

Proposition 3.2 contains in principal a preimage of B under φ_2. As the formulas show this is a quite complicated matrix. Unfortunately we were not able to find a simple preimage.

Example 2: We put $\omega = \sqrt{-10}$ and have

$$\mathcal{O}_{10} = \mathbb{Z} + \mathbb{Z}\omega \subseteq \mathbb{Q}(\omega).$$

$\mathcal{O}_{10}$ is the ring of integers in $\mathbb{Q}(\omega)$. First of all we describe a presentation of $\mathrm{SL}_2(\mathcal{O}_{10})$.

PROPOSITION 3.3: *The group* $\mathrm{SL}_2(\mathcal{O}_{10})$ *is generated by*

$$g_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; g_2 = \begin{pmatrix} 3\omega & -7 \\ 13 & 3\omega \end{pmatrix}, \; g_3 = \begin{pmatrix} 9 & 2\omega \\ -4\omega & 9 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \; g_5 = \begin{pmatrix} 1+\omega & -3 \\ 4 & -1+\omega \end{pmatrix}, \; g_6 = \begin{pmatrix} 1+\omega & 3 \\ -4 & -1+\omega \end{pmatrix}.$$

The quotient group $\mathrm{PSL}_2(\mathcal{O}_{10}) = \mathrm{SL}_2(\mathcal{O}_{10})/\langle g_1^2 \rangle$ *is defined by the relations:*

(1) $g_1^2 = (g_2 g_3^{-1})^2 = (g_1 g_2^{-1} g_3)^2 = (g_1 g_2 g_3^{-1})^2 = (g_6 g_3 g_2^{-1} g_5 g_1)^3 = 1,$
(2) $g_6 g_3 g_2^{-1} g_5 = g_6^{-1} g_2^{-1} g_3 g_5^{-1},$
(3) $g_3 g_6 g_1 g_5 g_3^{-1} g_5 g_1 g_6 = 1,$
(4) $g_4 g_6 g_1 g_5 g_4^{-1} g_5 g_1 g_6 = 1,$
(5) $g_4 g_6 g_3 g_2^{-1} g_5 g_4^{-1} = g_6 g_3 g_2^{-1} g_5.$

The group $U_2(\mathcal{O}_{10})$ *is generated as a normal subgroup by*

$$g_4 = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \; g_5^{-1} g_2 g_3^{-1} g_6^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; g_5 g_1 g_6 = \begin{pmatrix} 1+\omega & 2 \\ 5 & 1-\omega \end{pmatrix},$$
$$g_1^2 g_4 g_3^{-1} = \begin{pmatrix} -9 & 2\omega \\ 5\omega & 11 \end{pmatrix}.$$

This proposition is proved by analyzing an appropriate fundamental domain for the group $\mathrm{SL}_2(\mathcal{O}_{10})$ acting on the 3-dimensional hyperbolic space; see for example [Sw], [G,H,M]. We omit the details here.

PROPOSITION 3.4: *Let*

$$\varphi\colon \mathbb{Z}[x] \to \mathcal{O}_{10}$$

be the ring homomorphism which maps x to $\omega = \sqrt{-10}$. Then $\mathrm{SL}_2(\mathcal{O}_{10})$ is generated by the image under φ_2 of $\hat{E}_2(\mathbb{Z}[x])$ together with the images of the matrices:

$$h_1 = \begin{pmatrix} 6045x + 218 & 3224x^2 + 144x + 1 \\ 6975 & 3720x + 32 \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 5083x + 362 & 884x^2 + 77x + 1 \\ 6877 & 1196x + 19 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} 182x + 9 & -312x^2 + 5x + 1 \\ 98 & -168x + 11 \end{pmatrix}.$$

Proof: We have:

$$\begin{aligned} \varphi_2(h_1) = {} & g_1g_4g_1^{-1}g_4^{-1}g_2g_3^{-1}g_1g_5g_1g_6g_3g_2^{-1}g_5(g_6g_1g_5)^3g_3^{-1}g_2g_4^{-1} \\ & (g_5g_1g_6)^2g_3g_2^{-1}(g_5g_1g_6)^5g_2g_3^{-1}g_4g_3^{-1}g_4g_1, \\ \varphi(h_2) = {} & g_1g_4g_1^{-1}g_3^{-1}g_2g_6g_1^{-1}g_5g_4^{-1}g_6^{-1}g_1^{-1}g_3^{-1}g_2g_6g_1^{-1}g_2g_3^{-1}g_6^{-1}g_1^{-1} \\ & g_4^{-1}g_5g_1g_6g_3g_2^{-1}g_6^{-1}g_1^{-1}g_5^{-1}g_1g_6^{-1}g_2^{-1}g_3g_5^{-1}g_4g_3^{-1}g_2g_3^{-1}g_4g_1^2, \\ \varphi(h_3) = {} & g_1g_4^2g_1^{-1}g_4^{-1}g_5g_6g_3g_2^{-1}g_5g_3^{-1}g_2g_6g_1^{-1}g_5g_3^{-1}g_2g_6g_3g_2^{-1} \\ & g_5g_1^{-1}g_2^{-1}g_3g_4^{-2}g_1. \end{aligned}$$

The expressions were found by an algorithm which uses the tesselation of hyperbolic 3-space by a nice fundamental domain for $\mathrm{SL}_2(\mathcal{O}_{10})$. For the quotient group $\mathrm{SL}_2(\mathcal{O}_{10})/\hat{E}_2(\mathcal{O}_{10})$ we find a presentation:

$$\mathrm{SL}_2(\mathcal{O}_{10})/\hat{E}_2(\mathcal{O}_{10}) = \left\langle g_1, \ldots, g_6 \;\middle|\; \begin{array}{l} g_1 = g_4 = (g_2g_3^{-1})^2 = 1, \\ g_2g_3^{-1} = g_6g_5, \\ g_2g_3^{-1}g_2^{-1}g_3 = 1, \\ g_5g_6g_5^{-1}g_6^{-1} = 1. \end{array} \right\rangle.$$

A simple computation using the above expressions for $\varphi_2(h_i)$ in terms of the generators finishes the proof. ∎

The proof of Proposition 3.4 also shows that the pair of matrices h_2, h_3 can be mapped homomorphically to the generators of F_2.

Proof of Proposition 1.3: Using Proposition 3.3 we find that $\mathrm{SL}_2(\mathcal{O}_{10})/U_2(\mathcal{O}_{10})$ is infinite cyclic generated by the image of g_5. From the expression for $\varphi_2(h_3)$ in terms of the generators $g_1, \ldots, g_6$ we find

$$\varphi_2(h_3) \cdot U_2(\mathcal{O}_{10}) = g_5 \cdot U_2(\mathcal{O}_{10}).$$

This proves the result. ∎

Example 3: We put $\omega = \sqrt{-14}$ and

$$\mathcal{O}_{14} = \mathbb{Z} + \mathbb{Z}\omega \subseteq \mathbb{Q}(\omega).$$

The ring $\mathcal{O}_{14}$ is the ring of integers in $\mathbb{Q}(\omega)$. We have:

PROPOSITION 3.5: *The group* $SL_2(\mathcal{O}_{14})$ *is generated by the matrices:*

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ g_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ g_3 = \begin{pmatrix} 15 & -4\omega \\ 4\omega & 15 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix},\ g_5 = \begin{pmatrix} 5\omega & -11 \\ -19 & -3\omega \end{pmatrix},\ g_6 = \begin{pmatrix} 4+4\omega & -15-\omega \\ -15+\omega & 4-4\omega \end{pmatrix},$$
$$g_7 = \begin{pmatrix} 48-2\omega & 7-9\omega \\ 7+13\omega & 34+2\omega \end{pmatrix}.$$

The defining relations for $PSL_2(\mathcal{O}_{14}) = SL_2(\mathcal{O}_{14})/\langle g_1^2 \rangle$ *are*

(1) $g_1^2 = (g_1g_2)^3 = 1$,
(2) $g_1g_3g_1^{-1}g_3^{-1} = g_2g_4g_2^{-1}g_4^{-1} = 1$,
(3) $g_5g_6^{-1}g_7g_3^{-1}g_6g_5^{-1}g_3g_7^{-1} = 1$,
(4) $g_6g_4g_1g_3^{-1}g_6g_1g_6^{-1}g_3g_4^{-1}g_1g_6^{-1}g_1 = 1$,
(5) $g_1g_6g_1g_2g_6^{-1}g_1g_6^{-1}g_3g_1g_2^{-1}g_3^{-1}g_6 = 1$,
(6) $g_2^{-1}g_7^{-1}g_6g_2^{-1}g_1g_6^{-1}g_1g_5g_2g_6^{-1}g_7g_2g_1g_3^{-1}g_6g_1g_5^{-1}g_3 = 1$.

Again a proof is obtained by analyzing the action of $SL_2(\mathcal{O}_{14})$ on the 3-dimensional hyperbolic space.

PROPOSITION 3.6: *Let*

$$\varphi\colon \mathbb{Z}[x] \to \mathcal{O}_{14}$$

be the ring homomorphism which maps x to $\omega = \sqrt{-14}$. Then $SL_2(\mathcal{O}_{14})$ is generated by the image under φ_2 of $\hat{E}_2(\mathbb{Z}[x])$ together with the images of the matrices:

$$h_1 = \begin{pmatrix} 21x+10 & 6x^2+5x+1 \\ 49 & 14x+5 \end{pmatrix},$$
$$h_2 = \begin{pmatrix} 1320x+51 & -360x^2+12x+1 \\ 968 & -264x+19 \end{pmatrix},$$
$$h_3 = \begin{pmatrix} 182x+9 & -312x^2+5x+1 \\ 98 & -168x+11 \end{pmatrix},$$
$$h_4 = \begin{pmatrix} 21x+2 & -112x^2+x+1 \\ 9 & -48x+5 \end{pmatrix}.$$

Proof: The proof follows from the following expressions for the $\varphi_2(h_i)$ in terms of the generators:

$$\begin{aligned}
\varphi_2(h_1) &= g_1^{-1}g_3^{-1}g_5g_1^{-1}g_6g_1^{-1}g_2g_6^{-1}g_1g_5^{-1}g_6^2g_2^{-1}g_1g_6^{-1},\\
\varphi_2(h_2) &= g_1g_4^{-1}g_3^{-1}g_5g_1^{-1}g_2g_5^{-1}g_3g_2^{-1}g_3^{-1}g_6g_1g_5^{-1}g_1g_5g_6g_2^{-1}g_1g_6^{-1}g_3,\\
\varphi_2(h_3) &= g_1^{-1}g_4^{-2}g_1g_5^{-1}g_6g_2^{-1}g_1g_6^{-1}g_1g_5g_2^{-1}g_5^{-1}g_6^{-1}g_3g_2g_1^{-1}g_3^{-1}g_6g_1^{-1}g_6^2\\
&\quad g_2^{-1}g_1g_6^{-1}g_1g_4^2g_1,\\
\varphi_2(h_4) &= g_1^{-1}g_4^{-2}g_3^{-1}g_5g_6^{-1}g_7g_2g_4^5g_1^{-1}. \quad \blacksquare
\end{aligned}$$

COROLLARY 3.7: *Any 3 of the matrices $h_1, \ldots, h_4 \in \mathrm{SL}_2(\mathbb{Z}[x])$ given in Proposition 3.6 can be homomorphically mapped onto the free generators of a rank 3 free quotient of $\mathrm{SL}_2(\mathbb{Z}[x])/\hat{E}_2(\mathbb{Z}[x])$.*

Proof: Note that

$$\mathrm{SL}_2(\mathcal{O}_{14})/\hat{E}_2(\mathcal{O}_{14}) = \langle g_3, g_5, g_6, g_7 \mid g_5g_6^{-1}g_7g_3^{-1}g_6g_5^{-1}g_3g_7^{-1} = 1\rangle.$$

Take the quotient by $g_3g_7^{-1} = 1$ which is free on 3-generators and compute the images of $\varphi_2(h_1, \ldots, \varphi_2(h_4)$ in this quotient using the expressions in terms of the generators given in the proof of Proposition 6.4. $\blacksquare$

4. Free quotients of $\mathrm{SL}_2(k[x,y])$

To prove Theorems 1.4 and 1.6 we study the groups $\mathrm{SL}_2(\mathcal{O})$ where $\mathcal{O}$ are rings of integers in hyperelliptic function fields over the finite field k. We shall first of all fix some notation for the rest of this section.

For a polynomial $P \in k[x]$ we define

$$E_P = \begin{cases} y^2 - P(x) & \text{if } \mathrm{char} k \neq 2, \\ y^2 + y + P(x) & \text{if } \mathrm{char} k = 2. \end{cases} \tag{4.1}$$

The rings in question are

$$\mathcal{O}_P = k[x,y]/(E_P).$$

We let $k(x)$ be the rational function field over k and v the valuation at infinity of $k(x)$. That is, for polynomials R, S with $S \neq 0$ we have

$$v\Big(\frac{R}{S}\Big) = \deg S - \deg R.$$

Let K_0 be the completion of $k(x)$ with respect to v. We shall further assume that:

$$\text{(4.2)} \qquad P \text{ is not a square in } K_0.$$

Then

$$K = K_0[y]/(E_P)$$

is a ramified quadratic extension field of K_0. The field K is a complete local field with respect to the extension valuation w of v. Let A be the ring of integers in K.

We write X_K for the tree corresponding to the group $\mathrm{PGL}_2(K)$. For details see [Se1]. The embedding $\mathcal{O}_P \leq K$ gives rise to a discontinuous action of the group $\mathrm{SL}_2(\mathcal{O}_P)$ on X_K. The quotient $X_K/\mathrm{SL}_2(\mathcal{O}_P)$ is a graph of finite volume with finitely many cusps as described in [Se1].

Our assumption (4.2) also guarantees that $\mathcal{O}$ is an integral domain. We write

$$L_P$$

for its field of fractions. L_P is a field of transcendence degree 1, and $\mathcal{O}_P$ is its ring of integers, that is the intersection of all valuation rings in L_P for all valuations of L_P except w. We further write

$$\mathcal{C}_P$$

for the projective curve defined by the equation $E_l = 0$. We assume that

$$\text{(4.3)} \qquad \mathcal{C}_P \text{ is smooth.}$$

Let g be the genus of $\mathcal{C}_P$. We have

$$g = \begin{cases} \frac{\deg P - 1}{2} & \text{if } \deg P \text{ is odd,} \\ \frac{\deg P - 2}{2} & \text{if } \deg P \text{ is even.} \end{cases}$$

The ζ-function $\zeta_{\mathcal{C}_P}(s)$ of $\mathcal{C}_P$ can be expressed as

$$\zeta_{\mathcal{C}_P}(s) = \frac{Q(q^{-s})}{(1-g^{-s})(1-q^{1-s})}$$

with a polynomial Q of degree $2g$. Q can be factorized over $\mathbb{C}$ as:

$$Q(t) = \prod_{i=1}^{2g} (1 - \alpha_i t)$$

where the α_i all satisfy $|\alpha_i| = \sqrt{q}$. For all this see [W].

Proof of Theorem 1.4: Let $H^1(G,\mathbb{Q})$ stand for the first cohomology group with coefficients in $\mathbb{Q}$ of a group G. Under the assumptions (4.2) and (4.3) there is the formula:

$$\begin{aligned}\dim_{\mathbb{Q}} H^1(\mathrm{SL}_2(\mathcal{O}_P),\mathbb{Q}) &\geq \dim_{\mathbb{Q}} H^1(\mathrm{GL}_2(\mathcal{O}_P),\mathbb{Q}) \\ &= 1 + \frac{1}{q^2-1}\Big(Q(q) - \frac{q(q+1)}{2}Q(1) - \frac{q(q-1)}{2}Q(-1)\Big).\end{aligned}$$

For this see [G1, [G2]. It follows from the above that $Q(q)$ is roughly of size q^{3g} whereas both $Q(1)$ and $Q(-1)$ are only of size q^g. Since there are polynomials P so that $\mathcal{C}_P$ is smooth and has arbitrarily large genus, we may pick such a P so that

$$\dim_{\mathbb{Q}} H^1(\mathrm{SL}_2(\mathcal{O}_P),\mathbb{Q}) \geq m.$$

We may infer that the commutator quotient group of $\mathrm{SL}_2(\mathcal{O}_P)$ tensored with $\mathbb{Q}$ has dimension bigger than m. It follows from the structure theorem for groups acting discontinuously on trees ([Se1]) that there is a surjective homomorphism

$$\psi\colon \mathrm{SL}_2(\mathcal{O}_P) \to F_m.$$

Since the symplectic Whitehead group $K_1\mathrm{Sp}(\mathcal{O}_P)$ is trivial, the composition $\psi\circ\varphi_2$ maps $\mathrm{SL}_2(k[x,y])$ onto F_m. ∎

To obtain a proof of Theorem 1.6 we assume from now on that the characteristic of k is not 2 or 3. For an element $a \in k$ with $a \neq 0$ we consider from here on the cubic polynomial

$$P(x) = x^3 + a.$$

We write $\bar{y}$ for the image of y in K. We put

$$\pi = \frac{\bar{y}}{x^2}.$$

π is a uniformizing element in K, that is $\omega(\pi) - 1$. We wish to explicitly describe the action of $\mathrm{SL}_2(\mathcal{O}_P)$ on X_K. The tree X_K can be described as the graph corresponding to the classes of A-lattices in the vector space $V = K^2$. We write

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

for the standard basis of V. If u, v are elements of V we define

$$\langle u, v\rangle = Au + Av$$

to be the A-submodule of V generated by u, v. We have to introduce the following lattices in V.

Assume that μ is fixed and is not a square in k and $\alpha \in k$. We put

$$(4.4)\qquad \begin{aligned} \Lambda &= \langle e_1, e_2\rangle, \\ \Lambda_1 &= \langle e_1, \pi e_2\rangle, \\ \triangle_0 &= \langle e_1 + \pi e_2, \pi e_1\rangle, \\ \triangle_1 &= \langle e_1 + \mu\pi e_2, \pi e_1\rangle, \\ \Gamma_0 &= \langle e_1 + \pi e_2, \pi^2 e_1\rangle, \\ \Gamma_1 &= \langle e_1 + \mu\pi e_1, \pi^2 e_1\rangle, \\ \Lambda_0(\alpha) &= \langle (1+\alpha\pi^2)e_1 + \pi e_2, \pi^3 e_1\rangle, \\ \Lambda_1(\alpha) &= \langle (1+\alpha\pi^2)e_1 + \mu\pi e_2, \pi^3 e_1\rangle. \end{aligned}$$

If $\alpha \in k$ satisfies $P(\alpha) = 0$ we also define

$$(4.5)\qquad \Theta(\alpha) = \langle (1+\alpha\pi^2)e_1 + \pi e_2, \pi^4 e_1\rangle.$$

If $L \leq V$ is an A-lattice then we write $[L]$ for the class of L. $[L]$ is a vertex in X_k. Next we divide the set k into three subsets:

$$T_0 = \{\alpha \in k \mid P(\alpha) = 0\}, \quad T_1 = \{\alpha \in k \mid P(\alpha) \neq 0\},$$
$$T_2 = \{\alpha \in k \mid P(\alpha) \neq 0 \text{ and } P(\alpha) \text{ is a square in } k\}.$$

We then let G be the graph which is described in Fig. 1.

The vertices of G are the classes of certain of the lattices defined in (4.4) and (4.5). The $\alpha \in k$ are chosen so that

$$T_1 = \{\alpha_1, \ldots, \alpha_r\}$$

and the $\beta \in k$ satisfy

$$T_0 = \{\beta_1, \ldots, \beta_s\}.$$

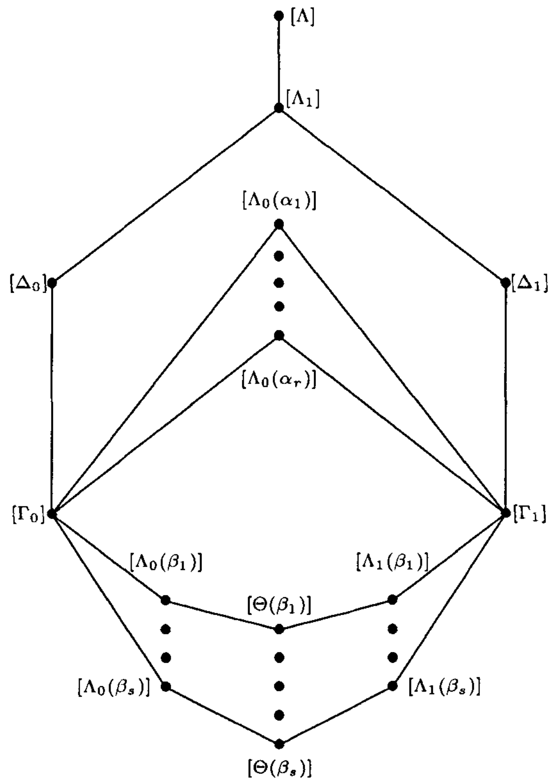


Fig. 1

We have:

PROPOSITION 4.1: *Let k be a finite field of characteristic not 2 and 3. Let $P = x^2 + a$ with $a \neq 0$. Then the following hold:*

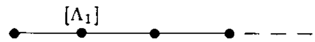
(i) *G is a subgraph of $\mathrm{SL}_2(\mathcal{O}_P) \setminus X_K$,*

(ii) *there is a retraction*

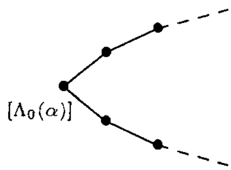
$$\beta\colon \mathrm{SL}_2(\mathcal{O}_P) \setminus X_K \to G.$$

This result is well known to specialists. See for example [St], [Ma]. In these papers the correspondence between classes of A-lattices and vector bundles over the smooth projective curve $\mathcal{C}_P$ is heavily used. We have proved Proposition 4.1 by an elementary classification of the classes of A-lattices up to $\mathrm{SL}_2(\mathcal{O})$ equivalence. We omit the details here. We remark that the graph $\mathrm{SL}_2(\mathcal{O}) \setminus X_K$ can be recovered from G by adding certain cusps. In the vertices $[\Theta(\beta)]$ the complete picture looks like

In the vertex $[\Lambda_1]$ a cusp has to be added also:



In the vertices $[\Lambda_0(\alpha)]$ for $\alpha \in T_2$ we have



The vertices shown in the last three pictures do not (except for $[\Theta(\beta)], [\Lambda_0(\alpha)]$ or $[\Lambda_1]$) belong to the graph G. This then gives $\mathrm{SL}_2(\mathcal{O}_P) \setminus X_K$ completely.

The elementary approach has the advantage that we can explicitly find a nice set of generators for $\mathrm{SL}_2(\mathcal{O}_P)$. Some of them play a role in the further arguments.

We can associate to an element $\gamma \in \mathrm{SL}_2(\mathcal{O})$ a closed path

$$W(\gamma)$$

in G be taking the image under β of the shortest path in X_K connecting $[\Lambda]$ to $[\gamma\Lambda]$. Next we have to compute the path $W(\gamma)$ for certain elements in $\mathrm{SL}_2(\mathcal{O}_P)$.

Definition 4.2: Let k be a finite field and $a \in k$ with $a \neq 0$. For elements $r,\ s \in k$ with

$$r^2 - as^2 = 1$$

we define

$$H_1(r,s) = \begin{pmatrix} r + s\bar{y} & -s^2 x \\ x^2 & r - s\bar{y} \end{pmatrix},$$

$$H_2(r,s) = \begin{pmatrix} r + (r^2 + s^2 a)x + (s + 2rsx)\bar{y} & -(r^2 + s^2 a) - s^2 x - 2rs^2 x^2 - 2rs\bar{y} \\ x^2 & r - x - s\bar{y} \end{pmatrix}.$$

Note that both $H_1(r,s)$ and $H_2(r,s)$ are in $\mathrm{SL}_2(\mathcal{O}_P)$. We have:

LEMMA 4.3: *Let k be a finite field of characteristic not 2 or 3 and $P = x^3 + a$ with $a \neq 0$. Let furthermore $r, s \in k$ with $r^2 - s^2 a = 1$. Then we have:*

(i) *The path $W(\gamma)$ of any unipotent element $\gamma \in \mathrm{SL}_2(\mathcal{O}_P)$ is 0-homotopic in G.*

(ii) *The path* $W(H_1(r,s))$ *is 0-homotopic in* G.

(iii) *Assume that* $2rs$ *is not a square, and that* $-s$ *is a square in* k, *then the path* $W(H_2(r,s))$ *starts with the vertices:*

$$[\Lambda], [\Lambda_1], [\triangle_2], [\Gamma_1], [\Lambda_1(\frac{1}{2r})], \ldots$$

and ends with the vertices:

$$\ldots, [\Lambda_0(s^{-2})], [\Gamma_0], [\triangle_0], [\Lambda_1], [\Lambda].$$

(iv) *Assume that* $2rs$ *is not a square and* $-s$ *is a square in* k, *then* $W(H_2(r,s))$ *is homotopic to the unique circle which starts and ends in* $[\Lambda]$ *and goes through* $[\Lambda_0(\alpha)]$ *where* α *is either* $(2r)^{-1}$ *of* s^{-2}.

Proof: (i) This is clear since a unipotent element has finite order, and the fundamental group of G is free.

(ii) Put

$$\gamma = \begin{pmatrix} x^{-2} & 0 \\ 0 & x^{-2} \end{pmatrix} H_2(r,s).$$

The vertices $[H_1(r,s)\Lambda]$ and $[\gamma\Lambda]$ coincide. The lattice $\gamma\Lambda$ is of "index" π^8 in Λ. Hence the path $W(\gamma)$ has length ≤ 8. We have:

$$\gamma\Lambda = \langle (rx^{-2} + s\pi)e_1 + e_2,\ -s^2x^{-1}e_1 + (rx^{-2} - s\pi)e_2 \rangle.$$

Constructing a series of sublattices between Λ and $\gamma\Lambda$ it is easy to see that in case $-s$ is a nonzero square the path of γ consists of the vertices

$$[\Lambda],\ [\Lambda_1],\ [\triangle_0],\ [\Gamma_0],\ [\Lambda_0(0)],\ [\Gamma_0],\ [\triangle_0],\ [\Lambda_1],\ [\Lambda]$$

together with the edges joining them.

In case $-s$ is a nonsquare the path of γ consists of the vertices:

$$[\Lambda],\ [\Lambda_1],\ [\triangle_1],\ [\Gamma_1],\ [\Lambda_1(0)],\ [\Gamma_1],\ [\triangle_1],\ [\Lambda_1],\ [\Lambda].$$

If $s = 0$ then $H_1(r,s)$ has finite order.

(iii) The computation needed is similar to the computation in (ii). We omit the details here.

(iv) This is an obvious consequence of (iii), since $W(H_2(r,s))$ has length ≤ 10. ∎

Lemma 4.3 is an example of many possible statements about $H_2(r,s)$. For example, there is a similar lemma describing the situation when $2rs$ is a square, but $-s$ is not.

We give now the connection of the elements $M(r,s)$ from the introduction and the matrices H_1, H_2.

LEMMA 4.4: *Let k be a finite field and $r, s \in k$. Assume that $r \neq 0, 1, -1$ and $s \neq 0$. Define*

$$a = (1 - r^2)s^{-2},$$

and $P = x^3 + a$. Let $\mathcal{O}_P = k[x,y]/(y^2 - P(x))$ and $\bar{M}(r,x)$ the image of $M(r,s)$ under the reduction homomorphism $k[x,y] \to \mathcal{O}_P$. Then there are $g_1, g_2 \in E_2(\mathcal{O}_P)$ with

$$\bar{M}(r,x) \cdot H_1(-r,-s) = g_1 H_2(r,s) g_2.$$

Proof: Consists of an elementary computation. ∎

Let now k be a finite field of characteristic not 2 or 3. And let $P = x^3 + a$ with $a \neq 0$. Then there is the homomorphism

$$\Psi\colon SL_2(\mathcal{O}_P) \to \pi_1(G, [\Lambda]) = F_{|k|}$$

which maps an element γ to the class of the loop $W(\gamma)$. Lemmas 4.3 and 4.4 make the proof of Proposition 1.6 clear.

In fact we have proved the following sharper result.

PROPOSITION 4.5: *Let k be a finite field of characteristic not 2 or 3 and let $a \in k$ with $a \neq 0$. Define*

$$D(a) = \{(r,s) \in k \mid r^2 - as^2 = 1;\ r \neq 0, 1, -1;\ 2rs \text{ is not a square in } k;\ -s \text{ is a square in } k\}.$$

For $p = (r,s) \in D(a)$ let $S(p)$ be the set

$$S(p) = \{(2r)^{-1}, s^{-2}\}.$$

If $p_1 = (r_1, s_1), \ldots, p_l = (r_l, s_l) \in D(a)$ are elements so that the sets $S(p_1), \ldots, S(p_l)$ are pairwise distinct, then there is a homomorphism

$$\varphi\colon SL_2(k[x,y]) \to F_l$$

which maps the $M(r_1,s_1), \ldots, M(r_l,s_l)$ to free generators of F_l.

Remark 4.6: In case $k = \mathbb{Z}/11\mathbb{Z}$ then the

$$p_1 = (6, 10), \quad p_2 = (8, 2)$$

lie in $D(2)$ and satisfy the requirements of Proposition 4.5.

5. Groups of unipotent matrices

Here we analyze the structure of factors groups

$$U_2(R)/\hat{E}_2(R),$$

where $R = \mathbb{Z}[x]$ or $k[x, y]$. We start off by the following lemma which follows easily from the fact that $A[x]$ is a unique factorization domain whenever A is a principal ideal domain.

LEMMA 5.1: *Let A be a principal ideal domain. Let $g \in \mathrm{SL}_2(A[x])$ be a unipotent element, then there are P, Q, $R \in A[x]$ with*

$$g = \begin{pmatrix} 1 - PQR & P^2R \\ -Q^2R & 1 + PQR \end{pmatrix}.$$

Next we consider the group homomorphism $\hat{E}_2(R) \to \hat{E}_2(S)$ induced by a surjective ring homomorphism $R \to S$.

LEMMA 5.2: *Let φ: $R \to S$ be a surjective ring homomorphism. Then $\hat{E}_2(S)$ is in the image of the induced map*

$$U_2(R) \to U_2(S).$$

Proof: $\hat{E}_2(S)$ is generated by the

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 - acr & a^2r \\ -c^2r & 1 + acr \end{pmatrix}$$

with $r \in S$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(S).$$

Taking any primages $\tilde{a}$, $\tilde{c}$, $\tilde{r} \in R$ of a, c, r we find that

$$\begin{pmatrix} 1 - \tilde{a}\tilde{c}\tilde{r} & \tilde{a}^2\tilde{r} \\ -\tilde{c}^2\tilde{r} & 1 + \tilde{a}\tilde{c}\tilde{r} \end{pmatrix}$$

is a primage of g. ∎

Under certain hypotheses we are able to control the homomorphisms $U_2(R) \to U_2(S)$ induced by ring homomorphisms $R \to S$.

LEMMA 5.3: *Let R be a domain and S a Dedekind domain. Let furthermore $\varphi\colon R \to S$ be a surjective ring homomorphism. Then the induced homomorphism*

$$U_2(R) \to U_2(S)$$

is surjective.

Proof: Let

$$g = \begin{pmatrix} 1+\rho & -\sigma \\ \tau & 1-\rho \end{pmatrix} \in U_2(S)$$

be a unipotent element with $\sigma \neq 0$. The ρ, σ, τ have to satisfy

$$\rho^2 = \sigma\tau.$$

We factorize the ideal

$$\sigma \cdot S = \mathcal{A}^2 \cdot \mathcal{B}$$

where the ideal $\mathcal{B}$ has no square factor. The above relation between ρ and σ implies that

$$\rho \cdot S = \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{I}$$

where $\mathcal{I}$ is some ideal of S. Find $b, d \in S$ so that

$$b^2 S + d^2 S = \mathcal{A}^2$$

so there are $r, s \in S$ with

$$\sigma = b^2 r + d^2 s.$$

The elements

$$a = \rho\sigma^{-1}b, \quad c = \rho\sigma^{-1}d$$

from the field of fractions of S are in fact in S by construction. Let $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{r}, \tilde{s}$ be primages in R of the corresponding elements in S. Define

$$h = \begin{pmatrix} 1+\tilde{a}\tilde{b}\tilde{r} & -\tilde{b}^2\tilde{r} \\ \tilde{a}^2\tilde{r} & 1-\tilde{a}\tilde{b}\tilde{r} \end{pmatrix} \begin{pmatrix} 1+\tilde{c}\tilde{d}\tilde{s} & -\tilde{d}^2\tilde{s} \\ \tilde{c}^2\tilde{s} & 1-\tilde{c}\tilde{d}\tilde{s} \end{pmatrix} \in U_2(R).$$

An easy computation shows that h is mapped to g. The rest of the proof is now obvious. ∎

Remark 5.4: The following example shows that not every unipotent element has a unipotent preimage. Let $S = \mathbb{Z} + \mathbb{Z}\sqrt{-14}$. S is the ring of integers in $\mathbb{Q}(\sqrt{-14})$ and a Dedekind domain. Let $R = \mathbb{Z}[x]$ and

$$\varphi: R \to S, \quad \varphi: x \mapsto \sqrt{-14}.$$

The element

$$g = \begin{pmatrix} 8 + 2\sqrt{-14} & -7 + \sqrt{-14} \\ -7 + 3\sqrt{-14} & -6 - 2\sqrt{-14} \end{pmatrix} \in \mathrm{SL}_2(S)$$

is unipotent. If g would have a unipotent preimage there would be (by Lemma 5.1) $p, q, r \in S$ with

$$pqr = -7 - 2\sqrt{-14}, \quad p^2 r = -7 + \sqrt{-14}, \quad q^2 r = 7 - 3\sqrt{-14}.$$

This is impossible since r would have to satisfy $r\bar{r} = 7$, with $\bar{r}$ being the complex conjugate of r. ∎

It is easily shown in general that although a unipotent element $g \in \mathrm{SL}_2(\mathcal{O}_d)$ may not have a unipotent preimage in $\mathrm{SL}_2(\mathbb{Z}[x])$, some power g^m has a unipotent preimage. The proof uses more residue considerations. In the above example, $m = 9$ works.

From the above we may conclude the following structural information about the groups $U_2(R)/\hat{E}_2(R)$ where $R = \mathbb{Z}[x]$ or $k[x, y]$.

PROPOSITION 5.5: *Let $m \geq 0$ be an integer. Then there is a surjective homomorphism*

$$U_2(\mathbb{Z}[x])/\hat{E}_2(\mathbb{Z}[x]) \to \mathbb{Z}^m.$$

Proof: Let $\mathcal{O}$ be the ring of algebraic integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$. Let h be its class number. The group $\mathrm{SL}_2(\mathcal{O})$ acts by fractional linear transformations on the projective line

$$\mathbb{P}^1(\mathbb{Q}(\sqrt{-d})).$$

Let $a_1, \ldots, a_h$ be representatives for the h orbits of $\mathrm{SL}_2(\mathcal{O})$ on $\mathbb{P}^1(\mathbb{Q}(\sqrt{-d}))$. The stabilizers

$$\mathrm{SL}_2(\mathcal{O})_{a_i} = U_i$$

are subgroups of $U_2(\mathcal{O})$. We may then consider the following homomorphisms induced by the inclusions:

$$\bigoplus_{i=1}^{h} U_i^{ab} \to U_2(\mathcal{O})^{ab} \to \mathrm{SL}_2(\mathcal{O})^{ab}.$$

Serre has proved ([Se3], section 3.5) that the image of the composite homomorphism has torsionfree rank h if d is bigger than or equal to 5. Clearly the torsionfree rank of the image of $\hat{E}_2(\mathcal{O})^{ab}$ in $SL_2(\mathcal{O})^{ab}$ is ≤ 1. Since $U_2(\mathcal{O})$ is finitely generated (it is generated by the U_i) it follows that

$$U_2(\mathcal{O})/\hat{E}_2(\mathcal{O}) \cdot [U_2(\mathcal{O}), U_2(\mathcal{O})]$$

has a free abelian quotient of rank $h-1$. Since h is not bounded as $d \to \infty$ the result follows from Lemma 5.3. ∎

PROPOSITION 5.6: *Let k be a finite field, $m \leq 0$ be an integer and V a k-vector space of countably infinite dimension. Then there is a surjective homomorphism*

$$U_2(k[x,y]) \to \underbrace{V * V * \cdots * V}_{m\text{-}times}$$

where $*$ stands for the free product of groups.

Proof: Let $\mathcal{O}$ be the ring of integers in some hyperelliptic function field over k. By [Se1], the group $U_2(\mathcal{O})$ is isomorphic to the free product of certain cuspidal groups. It is well known that the number of factors is not bounded if $\mathcal{O}$ varies. From Lemma 5.3 we conclude the result. ∎

6. The stable range of polynomial rings

We analyze here the exact range of the polynomial ring $A[x]$ for certain coefficient rings A.

An important result (see [Bl]) is that the stable range of R is less than or equal to $d+1$ if the maximal spectrum of R is a noetherian space of dimension d. Let A now be a locally principal ring. Since the Krull dimension of A is ≤ 1 we find that the stable range of $A[x]$ is either 2 or 3. This alternative can now be related to the structure of certain symplectic K-groups.

PROPOSITION 6.1: *Let A be a locally principal ring with $K\mathrm{Sp}_1(A) = 1$. Then the following two conditions are equivalent.*

(1) *$K_1\mathrm{SP}(B) = 1$ for any quotient ring B of $A[x]$.*

(2) *2 is the stable range of $A[x]$.*

The proof of Proposition 6.1 follows from the following elementary lemma.

LEMMA 6.2: *Let R be a commutative ring with 1 and $a \in R[x]$. For $p \in R[x]$, $\bar{p}$ denotes the image of p in $R[x]/(a)$. For a unimodular vector*

$$u = (p_1, p_2, a) \in Um_3(R[x])$$

the following are equivalent:

(1) *u is stable,*

(2) *every matrix*

$$\begin{pmatrix} \bar{p}_1 & \bar{p}_2 \\ * & * \end{pmatrix} \in \mathrm{SL}_2(R[x]/(a))$$

lies in the image of $\varphi_2\colon \mathrm{SL}_2(R[x]) \to \mathrm{SL}_2(R[x]/(a))$.

Finally we have to add a proof of Proposition 1.9.

Proof of Proposition 1.9: If B is a ring and $\mathcal{B}$ an ideal of B, we use the standard notation of [B,M,S], [V,S] of SK_1-theory. That is: $\mathrm{SL}_n(B,\mathcal{B})$ is the full congruence subgroup in $\mathrm{SL}_n(B)$ modulo $\mathcal{B}$, $E_n(\mathcal{B})$ is the group generated by the elementary matrices with parameter in $\mathcal{B}$, $E_n(B,\mathcal{B})$ is the normal closure in $E_n(B)$ of $E_n(\mathcal{B})$. The groups $\mathrm{SL}(B,\mathcal{B})$, $E(B)$, $E(B,\mathcal{B})$ are the limits of the finite dimensional groups. We put

$$\mathrm{SK}_1(B,\mathcal{B}) = \mathrm{SL}(B,\mathcal{B})/E(B,\mathcal{B}).$$

Let now $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field of discriminant $d \in \mathbb{Z}$, $d < 0$. Put

$$\omega = \frac{d + \sqrt{d}}{2}.$$

Then $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\omega$ is the full ring of integers in K. For a nonzero integer f put:

$$B = \mathbb{Z} + f\omega\mathbb{Z}.$$

The $f \cdot \mathcal{O}$ is an ideal of both $\mathcal{O}$ and B. It is in fact the conductor ideal of the order B. By Lemma 2.7 of [V,S] we find that $E(\mathcal{O}, f^2\mathcal{O}) \leq E(f\mathcal{O}) \leq E(B)$, hence inclusion defines a homomorphism

$$i\colon \mathrm{SK}_1(\mathcal{O}, f^2\mathcal{O}) \to \mathrm{SK}_1(B).$$

Since $\mathcal{O}/f^2\mathcal{O}$ is a finite ring we get $\mathrm{SK}_1(\mathcal{O}/f^2\mathcal{O}) = 0$ and $\mathrm{SL}(B) = \mathrm{SL}(\mathcal{O}, f^2\mathcal{O}) \cdot E(B)$, see [V,S] section 16. This implies that the homomorphism i is surjective.

Since $SK_1(\mathbb{Z}, f\mathbb{Z}) = 0$ we get that the kernel of i is contained in the image of $E(\mathcal{O}, f\mathcal{O})$. Let us now say that f has property $(*)$ if the homomorphism

$$j\colon SK_1(\mathcal{O}, f^2\mathcal{O}) \to SK_1(\mathcal{O}, f\mathcal{O})$$

given by inclusion is an isomorphism and if $SK_1(\mathcal{O}, f\mathcal{O}) \cong \mu_m$ where μ_m is the group of roots of units in K. It is proved in [B,M,S] that this is the case if f is sufficiently divisible by certain primes. From the above we conclude that i is an isomorphism if f has property $(*)$.

To simplify the discussion we assume now that $\mathcal{O}^* = \{\pm 1\}$ and hence μ_2 is the group of roots of units in k. Let

$$\left(\frac{a}{b}\right) \in \{\pm 1\}$$

be the quadratic residue symbol for $a, b \in \mathcal{O}$ with a prime to $2\mathcal{O}$. Then it is known that the map

$$\begin{pmatrix} a & b \\ * & * \end{pmatrix} \mapsto \left(\frac{b}{a}\right)$$

gives rise to an isomorphism

$$SK_1(\mathcal{O}, f^2\mathcal{O}) \to \{\pm 1\}$$

if f is sufficiently divisible by 2. In fact [B,M,S] gives an explicit formula for the necessary power of 2.

Take now $d = -20$, that is $K = \mathbb{Q}(\sqrt{-5})$ and $B = \mathbb{Z} + 2\mathbb{Z}\sqrt{-5}$. From [B,M,S] we conclude that $f = 2$ has property $(*)$ in this case. We also have that 12 and $21 + 4\sqrt{-5}$ generate the unit ideal in B and satisfy

$$\left(\frac{12}{21 + 4\sqrt{-5}}\right) = -1.$$

Hence every element of the form

$$g = \begin{pmatrix} 21 + 4\sqrt{-5} & 12 \\ * & * \end{pmatrix}$$

defines a nontrivial element in $SK_1(\mathbb{Z} + \mathbb{Z}\sqrt{-5}, 4(\mathbb{Z} + \mathbb{Z}\sqrt{-5}))$. Hence by the above argument the corresponding element is nontrivial in $SK_1(B)$ and then also in $KSp_1(B)$. We then consider the homomorphism

$$\varphi\colon \mathbb{Z}[x] \to B, \quad x \mapsto 2\sqrt{-5}.$$

From Lemma 6.2 we conclude that the unimodular vector given in Proposition 1.9 is not stable. ∎

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